On Convolution

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in June 1925 Heisenberg had hay fever

in Helgoland he apparently had an epiphany

Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen.

Von **W. Heisenberg** in Göttingen.

(Eingegangen am 29. Juli 1925.)

Zur Quantenmechanik.

Von **M. Born** und **P. Jordan** in Göttingen.

(Eingegangen am 27. September 1925.)

Zur Quantenmechanik. II.

Von **M. Born, W. Heisenberg** und **P. Jordan** in Göttingen.

(Eingegangen am 16. November 1925.)

the story goes that he invented matrix mechanics

Sei $x(t)$ durch $\mathfrak{A}$, $y(t)$ durch $\mathfrak{B}$ charakterisiert, so ergibt sich als Darstellung von $x(t) \,.\, y(t)$:

Klassisch:

$$\mathfrak{C}_\beta(n) = \sum_{-\infty}^{+\infty}{}^{\alpha} \, \mathfrak{A}_\alpha(n)\, \mathfrak{B}_{\beta-\alpha}(n).$$

Quantentheoretisch:

$$\mathfrak{C}(n, n-\beta) = \sum_{-\infty}^{+\infty}{}^{\alpha} \, \mathfrak{A}(n, n-\alpha)\, \mathfrak{B}(n-\alpha, n-\beta).$$

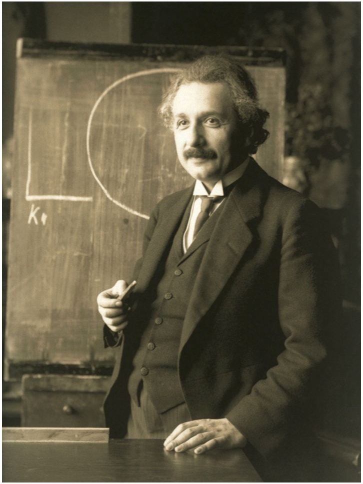
Während klassisch $x(t) \,.\, y(t)$ stets gleich $y(t)\, x(t)$ wird, braucht dies in der Quantentheorie im allgemeinen nicht der Fall zu sein. — In speziellen Fällen, z. B. bei der Bildung von $x(t) \,.\, x(t)^2$, tritt diese Schwierigkeit nicht auf.

... but some say he invented a convolution algebra

It is by a fundamental calling into question of classical mechanics that Heisenberg arrived at this goal and went well beyond his predecessors. This questioning of classical mechanics runs approximately as follows: in the classical model, the algebra of observable physical quantities can be directly read from the *group* Γ of emitted frequencies; it is the convolution algebra of this group of frequencies. Since Γ is a commutative group, the convolution algebra is commutative. Now, in reality one is not dealing with a group of frequencies but rather, due to the Ritz-Rydberg combination principle, with a groupoid $\Delta = \{(i,j);\ i,j \in I\}$ having the composition rule $(i,j)\cdot(j,k) = (i,k)$. The convolution algebra still has meaning when one passes from a group to a groupoid, and the convolution algebra of the groupoid Δ is none other than *the algebra of matrices* since the convolution product may be written

$$(ab)_{(i,k)} = \sum_j a_{(i,j)} b_{(j,k)} ,$$

[Connes: Noncommutative Geometry]



"In Göttingen glauben sie daran (ich nicht)."

so what is convolution?

Group Algebras

The convolution product. We shall now give examples of rings whose product is given by what is called convolution. Let G be a group and let K be a field. Denote by $K[G]$ the set of all formal linear combinations $\alpha = \sum a_x x$ with $x \in G$ and $a_x \in K$, such that all but a finite number of a_x are equal to 0. (See §3, and also Chapter III, §4.) If $\beta = \sum b_x x \in K[G]$, then one can define the product

$$\alpha\beta = \sum_{x \in G} \sum_{y \in G} a_x b_y xy = \sum_{z \in G} \left(\sum_{xy=z} a_x b_y \right) z.$$

With this product, the **group ring** $K[G]$ is a ring, which will be studied extensively in Chapter XVIII when G is a finite group. Note that $K[G]$ is commutative if and only if G is commutative. The second sum on the right above defines what is called a **convolution product**. If f, g are two functions on a group G, we define their **convolution** $f * g$ by

$$(f * g)(z) = \sum_{xy=z} f(x)g(y).$$

[Lang: Algebra]

this generalises to groupoids

Monoid Algebras

Let A be a commutative ring. Let G be a monoid, written multiplicatively.

Let $A[G]$ be the set of all maps $\alpha: G \to A$ such that $\alpha(x) = 0$ for almost all $x \in G$. We define addition in $A[G]$ to be the ordinary addition of mappings into an abelian (additive) group. If α, $\beta \in A[G]$, we define their product $\alpha\beta$ by the rule

$$(\alpha\beta)(z) = \sum_{xy=z} \alpha(x)\beta(y).$$

[Lang: Algebra]

and this to categories

Convolution

$$(f * g)(x) = \sum_{x=yz} f(y)g(z)$$

but why should computer scientists care?

Weighted Languages

$$(f * g)(x) = \sum_{x=y\cdot z} f(y) \cdot g(z)$$

$f, g : \Sigma^* \to S$ are formal power series

S is semiring, words are finitely decomposable

language theory à la Schützenberger

Languages

$$(f * g)(x) = \sum_{x = y \cdot z} f(y) \wedge g(z)$$

$f, g : \Sigma^* \to 2$

convolution is language product

Matrices

$$(f * g)(i,j) = \sum_k f(i,k) \cdot g(k,j)$$

$f, g : I \times I \to R$

$(i,j) = (i,k) \cdot (l,j)$ if $k = l$ (pair groupoid)

convolution is matrix product

Relations

$$(f * g)(i,j) = \bigvee_k f(i,k) \wedge g(k,j)$$

$f, g : I \times I \to 2$

convolution is relational composition

Fuzzy Relations

$$(f * g)(i,j) = \bigvee_k f(i,k) \cdot g(k,j)$$

$f, g : I \times I \to Q$ for quantale Q

fuzzy logic à la Goguen

Incidence Algebras

$$(f * g)(i,j) = \sum_k f(i,k) \cdot g(k,j)$$

$f, g : (I, \leq) \to R$ for locally finite poset category P

$(i,j) = (i,k) \cdot (l,j)$ if $k = l$

(i,j) means $i \leq j$

combinatorics à la Rota

Interval Temporal Logics

$$(f * g)(i,j) = \bigvee_k f(i,k) \wedge g(k,j)$$

$f, g : (I, \leq) \to 2$ for linear poset category $(I, \leq)$

convolution is chop modality

Separation Logic

$$(f * g)(\eta) = \bigvee_{\eta = \eta' \oplus \eta''} f(\eta') \wedge g(\eta'')$$

$f, g : (Loc \rightharpoonup Val) \to 2$

$\eta \oplus \eta' = \eta \cup \eta'$ if $dom(\eta) \cap dom(\eta') = \emptyset$

heaplets form partial abelian monoid (with single unit)

convolution is separating conjunction

Lambek Calculus

$$(f * g)(x) = \bigvee_{R(x,y,z)} f(y) \wedge g(z)$$

$f, g : X \to 2$

$R(x, y, z) \subseteq X \times X \times X$ is ternary Kripke frame

convolution is binary modality

Summary

convolution carries algebraic structure in each example

typically:

> *if* (X, R) *is a relational structure and* A *a suitable algebra then convolution algebra* A^X *forms same type of algebra, its composition is* $*$

we are repeating similar constructions!

can we unify/explain?

Relational Convolution

$$(f * g)(x) = \bigvee_{R(x,y,z)} f(y) \cdot g(z)$$

$f, g : X \to Q$

$R \subseteq X \times X \times X$

quantale $(Q, \leq, \cdot, 1)$:

- $(Q, \leq)$ is complete lattice
- $(Q, \cdot, 1)$ is monoid
- composition preserves sups in both arguments

Convolution as a Binary Modality

Q^X is complete lattice

2^X is even complete atomic boolean algebra

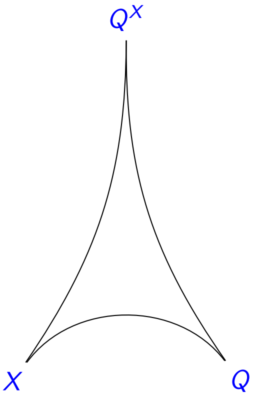
convolution $*$ is a binary modality on Q^X or 2^X

R is the corresponding/dual ternary Kripke frame

X and 2^X are related by Jónsson-Tarki duality
for boolean algebras with operators

what are the correspondences?

Modal Correspondence Triangle



Ternary Relations vs Multioperations

$$\mathcal{P}(X \times X \times X) \cong X \to X \to X \to 2 \cong \; X \times X \to \mathcal{P}X$$

ternary relations are multioperations $X \times X \to \mathcal{P}X$:

$$R(x, y, z) \Leftrightarrow x \in y \odot z$$

we lift to $\odot : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$

$$A \odot B = \bigcup \{x \odot y \mid x \in A, y \in B\}$$

ℓr-Multisemigroups

$(X, \odot, \ell, r)$ is an ℓr-multtsemigroup if

- $x \odot (y \odot z) = (x \odot y) \odot z$
- $x \odot y \neq \emptyset \Rightarrow r(x) = \ell(y)$ and $\ell(x) \odot x = \{x\} = x \odot r(x)$

it is a partial ℓr-semigroup if $|x \odot y| \leq 1$

it is local if $x \odot y \neq \emptyset \Leftrightarrow r(x) = \ell(y)$

in pair groupoid

- $(i, j) \odot ((j, k) \odot (k, l)) = \{(i, l)\} = ((i, j) \odot (j, k)) \odot (k, l)$
- $\ell(i, j) = (i, i)$, $r(i, j) = (j, j)$
- $(i, j) \odot (k, l) \neq \emptyset \Leftrightarrow r(i, j) = \ell(k, l)$

all our examples are based on ℓr-multisemigroups

ℓr-Multisemigroups

local partial ℓr-sgs are precisely (small object-free) categories

a groupoid is local partial ℓr-sg X in which each $x \in X$ has inverse x^{-1} with $x \odot x^{-1} = \{\ell(x)\}$ and $x^{-1} \odot x = \{r(x)\}$

in pair groupoid $(i, j)^{-1} = (j, i)$

Correspondences

if X is an ℓr-msg and Q a quantale, then Q^X is a quantale with

$$id_E(x) = \begin{cases} 1 & \text{if } x \in E \\ \perp & \text{otherwise} \end{cases} \qquad \text{where } E = \{x \mid \ell(x) = x\}$$

if Q^X, Q are quantales and $1 \neq \perp$ in Q, then X is an ℓr-msg

if Q^X is a quantale and X an ℓr-msg "with enough elements", then Q is a quantale

Finite Decomposability

an ℓr-msg is finitely decomposable if the following fibre is finite

$$\odot^{-1}(x) = \{(y, z) \mid x \in y \odot z\}$$

quantales can then be replaced by semirings

the correspondences still hold

Further Examples

path categories over digraphs $s, t : E \to V$ form local partial ℓr-sgs

- paths are $(v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n) : v_1 \to v_n$
- $\odot$ glues paths at their ends

they lift to path quantales

path cats over one vertex and n arrows give us words/weighted languages

heaplets are non-local partial ℓr-sgs
they lift to quantitative assertion quantales of separation logic

shuffle of words is proper local ℓr-mgs
it lifts to weighted shuffle quantales/semirings

paths $f : [0, 1] \to X$ in topology yield local partial ℓr-magma
it lifts only to prequantale

Extension: Quantitative Concurrent Quantales

a concurrent quantale is formed by quantales $(Q, \leq, \cdot, 1)$, $(Q, \leq, \|, 1)$ that satisfy

$$(w \parallel x) \cdot (y \parallel z) \leq (w \cdot y) \parallel (x \cdot z)$$

an interchange ℓr-msg is formed by "ℓr-msgs" $(X, \odot, 1)$, $(X, \otimes, 1)$ that satisfy

$$(w \otimes x) \odot (y \otimes z) \subseteq (w \odot y) \otimes (x \odot z)$$

we get the usual correspondence triangle

examples are weighted shuffle and graph/pomset languages

Extension: Quantitative Modal Quantales

a modal quantale is a quantale with comain/codomain maps satisfying

$$d(x) \cdot x = x \qquad d(x \cdot y) = d(x \cdot d(y)) \qquad d(x) \leq 1$$
$$d(\bot) = \bot \qquad d(x \vee y) = d(x) \vee d(y)$$

opposite axioms for r and $d \circ r = r$, $r \circ d = d$

modal quantales are algebraic relatives of dynamic logics
($|x\rangle d(y) = d(x \cdot d(y))$ etc, boxes are upper adjoints of diamonds)

we get the usual correspondence triangle

- $D(f)(x) = \bigvee_y d(f(y)) \cdot \delta_{\ell(y)}(x)$ and $R = \bigvee_y f(r(y)) \cdot \delta_{r(y)}(x)$
- ℓ lifts to D and r to R

examples are quantitative dynamic logics over categories and beyond

Extension: Girard Quantales

an effect algebra is a partial ℓr-sg $(X, \oplus, 0)$ with orthosupplement satisfying

- $x \oplus x^{\perp} = 0^{\perp}$
- $x \oplus 0^{\perp} \neq \emptyset$ implies $x = 0$

a commutative Girard quantale is quantale Q with dualising element d satisfying $x\backslash d\backslash d = x$ for all $x \in Q$

there is corresponding pair for X and $\mathcal{P}X$ with $\Delta = X - \{0^{\perp}\}$

this links algebras of effects in quantum mechanics with phase semantics of linear logic

... which brings us back to physics!

REVIEWS OF
MODERN PHYSICS

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Group Algebra, Convolution Algebra, and Applications to Quantum Mechanics*

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LIE GROUP CONVOLUTION ALGEBRAS AS DEFORMATION QUANTIZATIONS OF LINEAR POISSON STRUCTURES

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Lectures on the Geometry of Quantization

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SCHWINGER'S PICTURE OF QUANTUM MECHANICS: GROUPOIDS[1]

F.M. CIAGLIA, G. MARMO AND A. IBORT

A GROUPOID APPROACH TO QUANTIZATION

ELI HAWKINS