Coherence in higher-dimensional Kleene algebras

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Objective

- Main objective : provide an algebraic formulation of **coherence** in the context of **rewriting systems** and other formal calculi.
- Given a set expressions, such as $(1+1)$, and a set of permitted calculations, such as $(1+1 \to 2)$, called **reductions**, we consider certain properties expressing their compatibility. The uniqueness of an output is such a property, called **confluence**.
- In general, there are many ways to compute such an output. **Higher-dimensional rewriting** takes this into account, and provides constructive methods for specifying how to reduce an expression [2].
- **Kleene algebras** provide an algebraic context for expressing calculatory properties [1, 4]. These properties can then be specified and checked formally, replacing deductions by calculations in the algebra.
- **Higher-dimensional Kleene algebras** combine these properties, providing the formal algebraic context for higher-dimensional coherence.

Relation algebras

Let $\mathcal{R}(X)$ denote the set of binary relations on a set X. Given $R, S \in \mathcal{R}(X)$, we consider the following operations :

i) **Union** : denoted by $R \cup S$, with unit the empty set $\emptyset$,

ii) **Composition** : given by

$$R\,;S := \{\, (x,z) \in X \times X \mid \exists y \in X,\ (x,y) \in R \text{ and } (y,z) \in S \,\}.$$

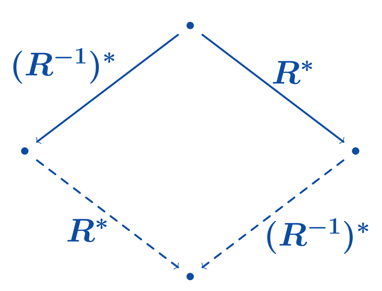
iii) **Reflexive, transitive closure** of R, denoted by R^*,

iv) **Inverse** of R, denoted by R^{-1}.

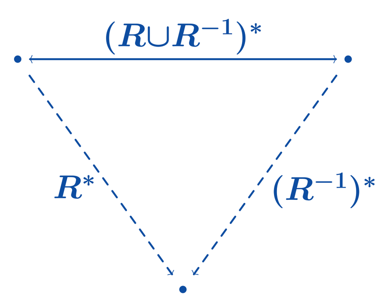
$(\mathcal{P}(X \times X), \cup, \emptyset, \ ;\ , \Delta, (-)^*)$ is the **full relation algebra** on X.

In this context, we express two coherence properties of interest. We say that a relation R is **confluent** (resp. **Church-Rosser**) when

$$(R^{-1})^*; R^* \subseteq R^*; (R^{-1})^* \qquad (\text{resp. } (R \cup R^{-1})^* \subseteq R^*; (R^{-1})^*)$$



Confluent



Church Rosser

The **Church-Rosser theorem** states that these properties are equivalent.

This result is generalised in [4] to the setting of Kleene algebras.

A **Kleene algebra** (**KA**) is an **idempotent semiring** $(K, +, 0, \cdot, 1)$ equipped with a map $(-)^* : K \to K$, the **Kleene star**, and operation which expresses **iteration**, *e.g.* reflexive, transitive closure in the context of relations.

Kleene algebras provide an abstract context for the Church-Rosser theorem :

Theorem (Church-Rosser in KA [4])

Let K be a Kleene algebra, and $a, b \in K$. The following equivalence holds in K :

$$b^* \cdot a^* \leq a^* \cdot b^* \qquad \Leftrightarrow \qquad (a+b)^* \leq a^* \cdot b^*.$$

The relational result is a corollary, when K is a relational algebra, and $b = a^{-1}$.

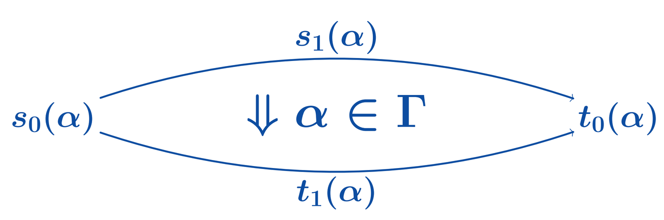
Higher dimensional rewriting

A **polygraph** Σ is a pair of sets (Σ_0, Σ_1) with maps $s_0, t_0 : \Sigma_1 \to \Sigma_0$, **0-source** and **0-target**. Σ_0 is the set of expressions and Σ_1 that of reductions :

$$\Sigma_0 \ni e_1 = s_0(u) \xrightarrow{\ u \in \Sigma_1\ } t_0(u) = e_2 \in \Sigma_0$$

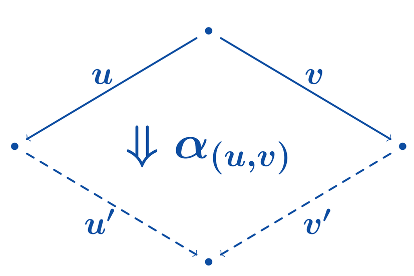
The **free category** (resp. **groupoid**) **generated** by Σ is denoted by Σ^* (resp. $\Sigma^\top$). This is the **reflexive transitive closure** (resp. **equivalence**) of Σ.

A **cellular extension** of Σ is a set Γ of **2-dimensional cells**, along with with maps $s_1, t_1 : \Gamma \to \Sigma^\top$, **1-source** and **1-target**, as in the adjacent diagram. These relate **parallel** reductions.



The free 2-category generated by Γ, *i.e.* the 2-cells obtained by glueing elements of Γ and their inverses along shared borders, is denoted by $\Sigma^\top[\Gamma]$.

A cellular extension Γ **fills confluences** if for every **branching** (u, v) of 1-cells of Σ^*, there exists a **confluence** (u', v') and a 2-cell $\alpha_{(u,v)}$ of $\Sigma^\top[\Gamma]$ as in the adjacent **confluence diagram**.

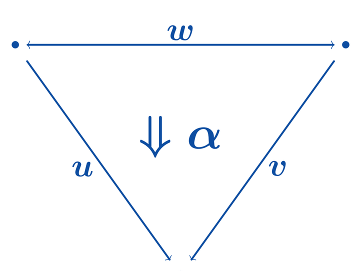


Theorem (Constructive Church-Rosser)

Let Γ be a cellular extension of Σ which fills confluences.

For any arrow w in $\Sigma^\top$, there exists a confluence (u, v) and a 2-cell $\alpha \in \Sigma^\top[\Gamma]$ such that

$$s_1(\alpha) = w \quad \text{and} \quad t_1(\alpha) = uv^{-1}$$



Lifting to sets of cells

- Using 2-dimensional cells to **fill holes** created by confluence diagrams has the advantage of tracking the different possible reductions while relating them.
- However, from a formal point of view, using polygraphs can be difficult.
- **Kleene algebras** can be utilised to formalise **coherence proofs** such as the constructive Church-Rosser theorem.
- Instead of looking at individual cells or diagrams, we consider **sets of cells** and **sets of diagrams**.

Modal 2-Kleene algebras

We introduce the algebraic context for higher-dimensional coherence.

A **2-modal Kleene algebra** (**2-MKA**) is a tuple $(K, +, 0, \odot_i, 1_i, (-)^{*_i})_{0 \leq i \leq 1}$ such that

i) $(K, +, 0, \odot_i, 1_i, (-)^{*_i})$ is a Kleene algebra for $0 \leq i \leq 1$,

ii) For $A, A', B, B' \in K$,

$$(A \odot_1 A') \odot_0 (B \odot_1 B') \leq (A \odot_0 B) \odot_1 (A' \odot_0 B') \quad \text{and} \quad 1_1 \odot_0 1_1 = 1_1.$$

Additionally, we have **domain** and **range** maps, $d_i, r_i : K \to K$ satisfying axioms characterising domain and range in the relational case.

The **domain algebra of dimension i** is the set $d_i(K) = r_i(K)$, and is denoted by K_i. Restricting $+$ and $\odot_i$ to K_i, we obtain a distributive lattice.

This structure is equipped with **modal i-diamond** operators defined via domain and range. For any $0 \leq i \leq 1$, $A \in K$ and $\phi \in K_i$, define

$$|A\rangle_i(\phi) = d_i(A \odot_i \phi) \qquad \langle A|_i(\phi) = r_i(\phi \odot_i A).$$

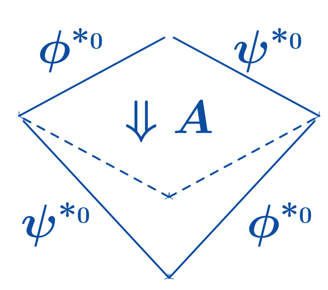
These are operators on the domain algebra in the sense of [3].

Confluence filler in modal 2-MKA

Let K be a globular 2-MKA.

$A \in K$ is a **confluence filler** for $(\phi, \psi) \in K_1 \times K_1$ if

$$d_1(A) = \phi^{*_0} \odot_0 \psi^{*_0} \quad \text{and} \quad r_j(A) \leq \psi^{*_0} \odot_0 \phi^{*_0}.$$



Main results

In the algebraic context established above, we have a Church-Rosser theorem :

Theorem (Coherent Church-Rosser in globular n-MKA)

Let K be a globular 2-MKA. Given $\phi, \psi \in K_1$ and any confluence filler $A \in K$ for (ϕ, ψ), we have

$$|\hat{A}^{*_1}\rangle_1(\psi^{*_0}\phi^{*_0}) \geq (\phi + \psi)^{*_0},$$

where $\hat{A} = (\phi+\psi)^{*_0} \odot_0 A \odot_0 (\phi+\psi)^{*_0}$.

Similarly to [1], we add a **Boolean structure** to the domain algebras, and thus characterise **termination** and **well-foundedness** in 2-MKA.
We obtain a generalisation of **Newman's lemma**, another classical rewriting theorem :

Theorem (Coherent Newman's lemma for globular Boolean 2-MKA)

Let K be a globular Boolean 2-MKA such that

i) $(K_0, +, 0, \odot_0, 1_0, \neg_0)$ *is a complete Boolean algebra,*

ii) *For all $\psi, \psi' \in K_1$ and every family $(p_\alpha)_{\alpha \in I} \subseteq K_0$, we have*

$$\psi \odot_0 sup_I(p_\alpha) \odot_0 \psi' = sup_I(\psi \odot_0 p_\alpha \odot_0 \psi').$$

Further, let $\phi, \psi \in K_1$ such that ψ terminates and ϕ is well-founded. If A is a ***local*** *confluence filler for (ϕ, ψ), we have*

$$|\hat{A}^{*_1}\rangle_1(\psi^{*_0}\phi^{*_0}) \geq \phi^{*_0}\psi^{*_0}.$$

Conclusions and perspectives

In summary, combining the formalism of Kleene algebras with the constructive approach given by higher dimensions, we obtain generalisations of the Church-Rosser theorem and Newman's lemma. A final coherence result, **Squier's theorem**, is the next step in our work.

All of the above results may be generalised to the context of **n-dimensional modal Kleene algebra**, a higher dimensional version of the structure described above.

References

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